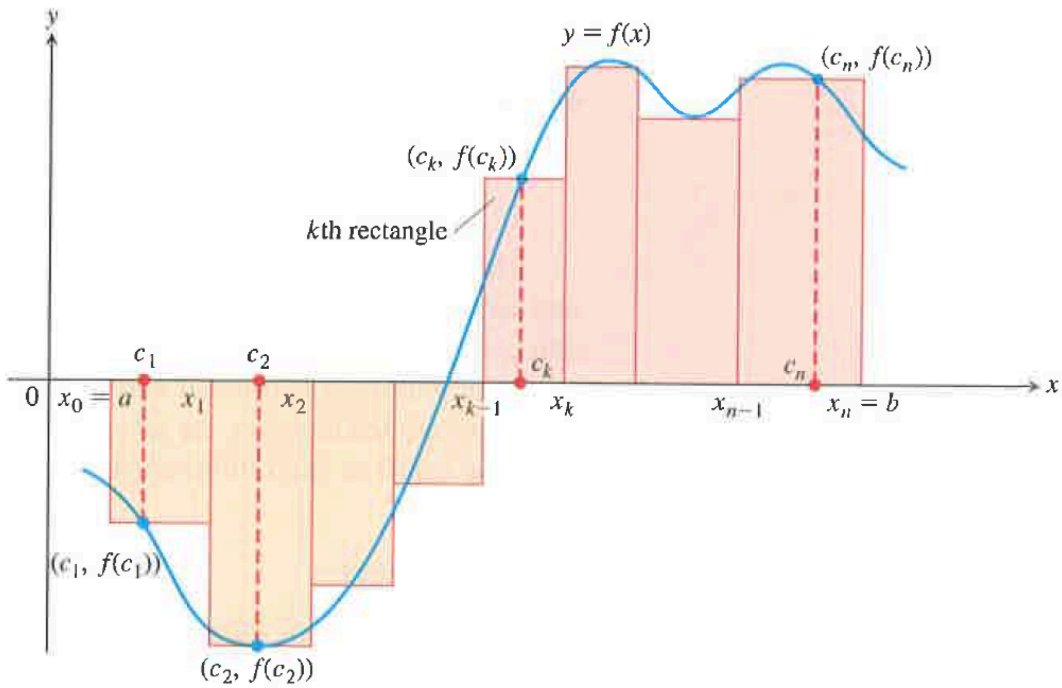
Riemann Sums, Definite Integral



How should we approximate with areas of rectangles?

1. We need to partition the interval $[a, b]$ into small subintervals.

2. We must then use the function f to determine the height of each rectangle and decide whether to count the area positively or negatively.

<u>Definition</u>

A partition of $[a, b]$ is a set of points

$\{x_0, x_1, x_2, x_3 \dots x_{k-1}, x_k, \dots, x_{n-1}, x_n\}$ such that

$a = x_0 < x_1 < x_2 < x_3 < \dots < x_{k-1} < x_k < \dots < x_{n-1} < x_n = b$.

Examples of partitions:

Our goal is to approximate areas with ever increasing degree of accuracy; so we will want our partitions to define a large number of subintervals with small width. If P is a partition of $[a,b]$ then determine how good the partition is by considering the length of the largest subinterval.

Some more notation:

Let $P = \{x_0, x_1, x_2, x_3 \ldots x_{k-1}, x_k, \ldots, x_{n-1}, x_n\}$ be a partition of $[a,b]$.

So $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{k-1} < x_k < \ldots < x_{n-1} < x_n = b$.

The subintervals are

$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{k-1}, x_k], \ldots, [x_{n-1}, x_n]$. Let

$\Delta x_1 = x_1 - x_0$, $\Delta x_2 = x_2 - x_1$, ... , $\Delta x_k = x_k - x_{k-1}$, ... , $\Delta x_n = x_n - x_{n-1}$

The length of the largest subinterval is equal to

$\max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_k, \ldots, \Delta x_n\}$

and is denoted by $\|P\|$, called the norm of the partition P.

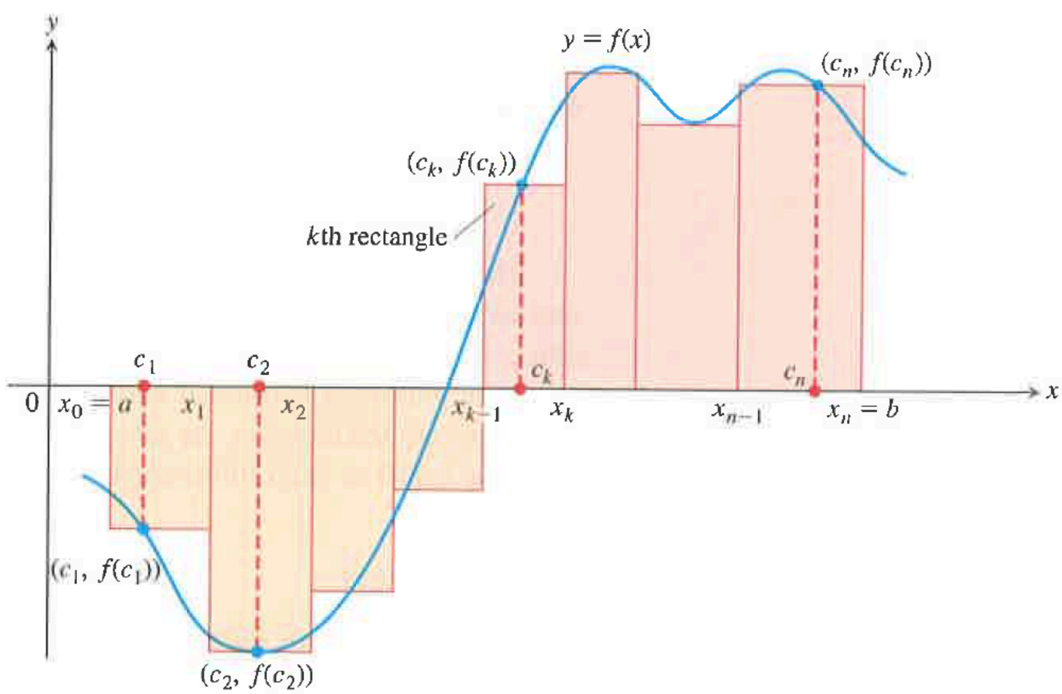
If we want our approximation to be accurate, then we want $\|P\|$ to be small (close to zero).

For the heights of the rectangles we will choose a point

c_k from each $[x_{k-1}, x_k]$ and evaluate $f(c_k)$.

We now obtain a Riemann sum:

$\sum_{k=1}^{n} f(c_k)\Delta_k$. What happens when $\|P\| \to 0$?



Definition

Suppose f is a continuous function on $[a,b]$. The definite integral of f over $[a,b]$ is

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta_k \quad \text{and is denoted by} \quad \int_a^b f(x)\,dx\,.$$

We read "the integral of f from a to b with respect to x".

Formally,

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta_k = L \qquad \text{means}$$

$$\textit{for each } \varepsilon>0\text{, there exists } \delta>0 \textit{ such that}$$

$$\left| \sum_{k=1}^{n} f(c_k)\Delta_k - L \right| < \varepsilon \;\; \textit{whenever} \; \|P\| < \delta$$

As long as the norm of the partition is small enough (norm less than delta), it doesn't matter what point you choose from each subinterval. The Riemann sum will be within epsilon of L.

TABLE 5.6 Rules satisfied by definite integrals

1. *Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A definition
2. *Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	A definition when $f(a)$ exists
3. *Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any constant k
4. *Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5. *Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6. *Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$.	
7. *Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
	$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special case)

Suppose f is a continuous function on $[a,b]$. The average value of f on $[a,b]$ can be computed in terms of a definite integral.

Average value of f on $[a,b]$ is equal to $\frac{1}{b-a}\int_a^b f(x)\,dx$

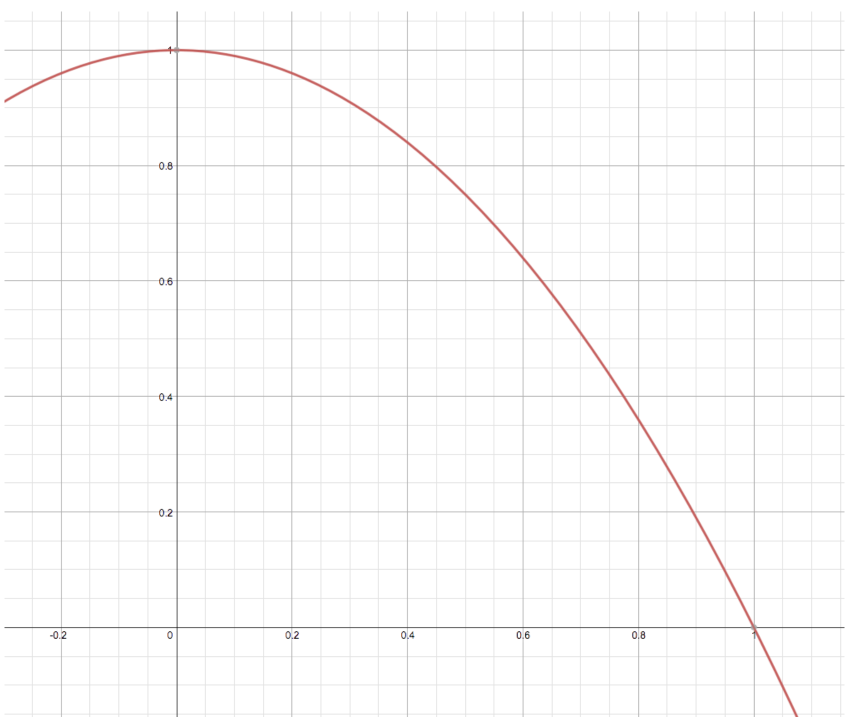
How do we compute $\int_a^b f(x)\,dx$?

We are going to show that if f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a) \quad \text{where } F \text{ is any}$$

antiderivative of f.

Let's find the area below the graph of $y = f(x) = 1 - x^2$ between $x=0$ *and* $x=1$.



$$\int_0^1 (1 - x^2)\,dx = \left(x - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{2}{3}$$

Fundamental Theorem of Calculus

I. Suppose f is a continuous function on $[a,b]$. Let

$F(x)=\int_a^x f(t)\,dt$ for $a \le x \le b$. Then $F'(x)=f(x)$ for each x.
Also note that $F(b)=\int_a^b f(x)\,dx$.

Proof:

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt}{h}$$

$$\lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}=\lim_{h\to 0}\frac{f(t_h)(h)}{h}=\lim_{h\to 0} f(t_h) \text{ where } x\le t_h \le x+h.$$

As $h\to 0$, $t_h \to x$, *so* $\lim_{h\to 0} f(t_h)=f(x)$ *and we get* $F'(x)=f(x)$.

II. Suppose f is a continuous function on $[a,b]$. Let $G(x)$ be any antiderivative of $f(x)$. Then

$$\int_a^b f(x)\,dx = G(x)\Big|_a^b = G(b)-G(a).$$

Proof.

If $F(x)=\int_a^x f(t)\,dt$ for $a \le x \le b$ then G and F differ by at most a constant and we have

$G(x)=F(x)+c$ for $a \le x \le b$. Now,

$G(a)=F(a)+c = 0+c = c$, and $G(b) = F(b)+c = F(b)+G(a)$

We now have $G(b)-G(a) = F(b) = \int_a^b f(x)\,dx$.

Evaluate the following definite integrals:

a) $\int_0^{\pi} \sin x \, dx$

b) $\int_0^{\pi/4} (3x^2 + 4\cos 2x)\, dx$

c) $\int_0^{3} t\, e^{t^2}\, dt$

d) $\int_0^{1} \frac{6}{1+4x^2}\, dx$

e) $\int_0^{1/2} \frac{7}{\sqrt{1-x^2}}\, dx$